# AN EXTENSION OF A THEOREM OF JEROSLOW AND KORTANEK †

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ABSTRACT

Given a semi-infinite system of linear inequalities, including strict inequalities, it is shown that if every finite subsystem has a solution in $R$, then the entire system has a solution in the ordered field $R(M)$ obtained by adjoining a transcendental greater than every real number.

## 1. Introduction

A semi-infinite system of linear inequalities is a system of infinitely many inequalities in finitely many variables, typically:

$$a_{i1}x_1 + \cdots + a_{in}x_n \geqq b_i, \qquad i \in J \tag{1}$$

where $a_{ij}$, $b_i \in R$ and $J$ is arbitrary.

The essential new feature of semi-infinite systems as compared with ordinary finite systems is the possible presence of sequences of inequalities $x_1 \geqq 1, x_1 \geqq 2, \cdots$.

This suggests that one might obtain results about such systems by considering solutions involving infinite elements. In [1], Jeroslow and Kortanek considered the polynomial ring $R[M]$ with a lexicographic ordering and proved that if every finite subsystem of (1) had a solution in which $x_1, \cdots, x_n \in R$, then the entire system had a solution in which $x_1, \cdots, x_n \in R[M]$. In this paper we show this result may be obtained using the Fourier-Dines-Motzkin elimination technique [2, pp. 11–20]. We also extend the result to systems including strict inequalities.

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## 2. The ring $R[M]$ and the field $R(M)$

As indicated, $R[M]$ consists of the ring of polynomials with real coefficients with this ordering: $P \geqq Q$ iff $\lim_{M\to\infty} P(M) - Q(M) \geqq 0$. It may be verified that this gives a commutative ordered ring without divisors of zero. The standard field-of-quotients construction may then be carried out to give an ordered field $R(M)$. Note that if $p_1, \cdots, p_n \in R(M)$ satisfy a system (1), then the sequence $p_1(k), \cdots, p_n(k)$, $k = 1, 2, \cdots$, is a weak solution to (1) in the sense of [3].

We shall study semi-infinite systems of inequalities which include strict inequalities:

$$
\begin{aligned}
&a_{i1}x_1 + \cdots a_{in}x_n \geqq b_i, \quad i \in J_1 \\
&a_{i1}x_1 + \cdots a_{in}x_n > b_i, \quad i \in J_2
\end{aligned}
\qquad J_1 \cap J_2 = \varnothing. \tag{2}
$$

The system $x_1 > 0$, $x_1 < \frac{1}{2}$, $x_1 < \frac{1}{3}, \cdots$ gives an example of a case where every finite subsystem has solutions yet the entire system has no solution in $R[M]$. Note, however, that $x = 1/M$ is a solution in $R(M)$. The following is our main result.

THEOREM 1. *If every finite subsystem of* (2) *has a solution in the reals then* (2) *has a solution in* $R(M)$ *of the form*

$$a_j M^j + \cdots + a_0 + \cdots + a_{-k} M^{-k}, \tag{3}$$

*where* $j + k \leqq n$.

Proof of the theorem is given in Section 4.

## 3. Shifting and Dedekind cuts in $R[M]$

For $k \geqq 1$, we define a linear operation, the $k$-shift, on $R[M]$ as follows: the $k$-shift of $M^j$ is $M^{j+1}$ for $j \geqq k$, and $M^j$ for $j < k$. If $p \in R[M]$ we denote the $k$-shift of $p$ by $p^{(k)}$ and if $A \subset R[M]$, $A^{(k)}$ is the set of $k$-shifts of members of $A$. Note that if $p_1, \cdots, p_n \in R[M]$ then

$$a_{i1}p_1^{(k)} + \cdots + a_{in}p_n^{(k)} = (a_{i1}p_1 + \cdots + a_{in}p_n)^{(k)}$$

so that if $p_1, \cdots, p_n$ satisfy (2) then so do $p_1^{(k)}, \cdots, p_n^{(k)}$. We shall use $A \geqq B$ to mean that every member of $A$ is greater than or equal to every member of $B$. (A Dedekind cut in $R[M]$.) It is possible to have $A \geqq B$ and yet have no in-between element $p \in R[M]$ with $p \leqq A$ and $p \geqq B$. Example: $A = \{\frac{1}{2}M, \frac{1}{3}M, \cdots\}$, $B = \{1, 2, \cdots\}$.

LEMMA 1. *Suppose* $A, B \subset R[M]$ *and that* $A \geqq B$. *Then either*

(i) *there is a* $p \leqq A$ *and* $\geqq B$ *or*

(ii) *there is a* $k \geqq 1$ *and* $p$ *such that* $p < A^{(k)}$ *and* $p > B^{(k)}$.

PROOF. It is sufficient to prove this for the case where every member of $A$ is greater than or equal to zero. The other case may be reduced to this one by interchanging $A$ and $B$ after multiplying their members by $-1$. We proceed by induction on the smallest degree of members of $A$. If $A$ has members of degree zero (that is, real numbers) we may take $p$ to be the greatest lower bound of such members and (i) holds. Suppose the theorem has been established when $A$ has members of degree $k-1$ and that $A$ has a member of degree $k$. Let $a$ be the greatest lower bound of the set of $r$ such that $A$ has a member of the form $rM^k +$ lower-degree terms.

*Case* 1. Neither $A$ nor $B$ has a member of the form $aM^k +$ lower terms. We set $p = aM^k$ and (i) holds.

*Case* 2. $A$ has a member of this form but $B$ does not. In this case $p = aM^{k+1} - M^k$ is less than $A^{(k)}$ and greater than $B^{(k)}$, so (ii) holds.

*Case* 3. $A$ has no such member but $B$ does. Similar to Case 2, with $p = aM^{k+1} + M^k$.

*Case* 4. Both $A$ and $B$ have members of the form $aM^k +$ lower terms. Form $A'$ and $B'$ by subtracting $aM^k$ from members of $A$ and $B$. We have $A' \geqq B'$ and $A'$ and $B'$ have members of degree $k-1$, so we may apply the induction hypothesis.

Q.E.D.

We also need a linear operation, the $k$-slide, defined on members of $R(M)$ of the form $a_1M^{-1} + \cdots + a_jM^{-j}$. The $k$-slide of $M^{-j}$ is $M^{-j-1}$ for $j \geqq k$, and $M^{-j}$ for $j < k$. This is a reflection of the $k$-shift. $k$-slides of solutions of (2) are also solutions. The result we need on $k$-slides is the following.

LEMMA 2. *Let $A$ and $B$ be subsets of $R(M)$ of the form $a_1M^{-1} + \cdots + a_kM^{-k}$, where $k \leqq n$ for every member of $A \cup B$. If $A \geqq B$ then*

(i) *there is a $p$ of the same form with $p \leqq A$ and $p \geqq B$; or*

(ii) *there are $p, k$ such that $p$ is less than the $k$-slide of $A$ and greater than the $k$-slide of $B$.*

PROOF. Form $A'$ and $B'$ by multiplying the members of $A$ and $B$ by $M^n$. Then by Lemma 1 there is a $p$ in $R[M]$ such that

(i) $p \leqq A'$ and $p \geqq B'$ or

(ii) $p < A'^{(j)}$ and $p > B'^{(j)}$. If (i) holds, we divide $p$ by $M^n$ to obtain a member of $R(M)$ which is less than or equal to $A$ and greater than or equal to $B$. If (ii) holds we divide $p$ by $M^{n+1}$ to obtain a member of $R(M)$ which is less than the $(n-j+1)$ slide of $A$ and greater than the $(n-j+1)$ slide of $B$. Q.E.D.

If a member of $R(M)$ has the form (3) we shall call $a_jM^j + \cdots + a_0$ the polynomial part and $a_{-1}M^{-1} + \cdots + a_{-k}M^{-k}$ the infinitesimal part. We may extend our definition of $k$-shift and $k$-slide to elements (3) by taking the $k$-shift of the polynomial part or the $k$-slide of the infinitesimal part. We denote the $k$-slide of $p \in R(M)$ by $p^{(-k)}$.

LEMMA 3. *Suppose $A, B \subset R(M)$ of the form* (3) *with $k \leqq n$ for every member of $A \cup B$. If $A \geqq B$ then either*

(i) *there is a $p$ of the form* (3) *with $p \leqq A$ and $p \geqq B$ or*

(ii) *there is a $k$-shift or $k$-slide and $p$ such that $p < A^{(k)}$ and $p > B^{(k)}$.*

PROOF. Clearly every polynomial part of a member of $A$ is greater than or equal to every polynomial part of a member of $B$, so we may apply Lemma 1. If (ii) holds then we obtain a $p \in R[M]$ which is less than $A^{(k)}$ and greater than $B^{(k)}$ for some $k$-shift. If (i) holds there is a $p \in R[M]$ which is less than or equal to the polynomial part of every member of $A$ and greater than or equal to the polynomial part of every member of $B$. If neither $A$ nor $B$ have members with polynomial part $p$, then $p$ is less than or equal to $A$ and greater than or equal to $B$. If one but not both of $A, B$ has a member with polynomial part $p$, then $p \pm M^{-1}$ is less than $A^{(-1)}$ and greater than $B^{(-1)}$ (recall that this is a 1-slide). Finally, if both $A$ and $B$ have members with polynomial part $p$, form $A'$ and $B'$ by subtracting $p$ from such members and apply Lemma 2.

## 4. Proof of Theorem 1

The argument proceeds by induction on the number of unknowns in (2). Our theorem is obvious for systems with one unknown, so we will be done if we can prove the following reduction step.

LEMMA 4. *The system of inequalities*

$$x_n \geqq a_{i1}x_1 + \cdots \ a_{jn-1}x_{n-1} + b_i, \qquad i \in k_1$$

$$x_n > a_{i1}x_1 + \cdots \ a_{in-1}x_{n-1} + b_i, \qquad i \in k_2$$

$$(2') \quad \begin{aligned} x_n &\leqq a_{i1}x_1 + \cdots a_{in-1}x_{n-1} + b_i, && i \in k_3 \\ x_n &< a_{i1}x_1 + \cdots a_{in-1}x_{n-1} + b_i, && i \in k_4 \\ & a_{i1}x_1 + \cdots a_{in-1}x_{n-1} \geqq b_i, && i \in k_5 \\ & a_{i1}x_1 + \cdots a_{in-1}x_{n-1} > b_i, && i \in k_6 \end{aligned}$$

*has a solution with $x_1, \cdots, x_n$ of the form* (3) *if the following system does*:

(4)

$$\begin{aligned} &a_{i1}x_1 + \cdots + a_{in-1}x_{n-1} + b_i < a_{j1}x_1 + \cdots + a_{jn-1}x_{n-1} + b_j, \ i \in k_2, \ j \in k_3 \cup k_4 \\ &a_{i1}x_1 + \cdots + a_{in-1}x_{n-1} + b_i < a_{j1}x_1 + \cdots + a_{jn-1}x_{n-1} + b_j, \ i \in k_1, \ j \in k_4 \\ &a_{i1}x_1 + \cdots + a_{in-1}x_{n-1} + b_i \leqq a_{j1}x_1 + \cdots + a_{jn-1}x_{n-1} + b_j, \ i \in k_1, \ j \in k_3 \\ &a_{i1}x_1 + \cdots + a_{in-1}x_{n-1} \geqq b_i, \quad i \in k_5 \\ &a_{i1}x_1 + \cdots + a_{in-1}x_{n-1} > b_i, \quad i \in k_6. \end{aligned}$$

Note that any system (2) may be written in an equivalent manner in the form (2′). Once this lemma is established it suffices to notice that if every finite subsystem of (2′) is consistent, then so is every finite subsystem of (4), which has $n - 1$ unknowns.

PROOF. Let $x_1, \cdots, x_{n-1}$ be a solution to (4). If $k_1 \cup k_2$ or $k_3 \cup k_4$ is empty we obtain a solution to (2′) by taking $x_n = \pm M^j$, for $j$ sufficiently large. Otherwise we let $A = \{a_{i1}x_1 + \cdots + a_{in-1}x_{n-1} + b_i \mid i \in k_3 \cup k_4\}$ and

$$B = \{a_{i1}x_1 + \cdots + a_{in-1}x_{n-1} + b_i \mid i \in k_1 \cup k_2\}$$

and apply Lemma 3. If (ii) holds, then $x_n = p$ and $x_1 = x_1^{(k)}, \cdots, x_{n-1} = x_{n-1}^{(k)}$ is a, solution to (2′). If (i) holds then $x_n = p$ and $x_1 = x_1, \cdots, x_{n-1} = x_{n-1}$ is a solution to (2′) with $>$ replaced by $\geqq$ and $x_n = p \pm M^{-j}$ is a solution to (2′) for sufficiently large $j$. This completes the proof.

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